Math Logic: Model Theory & Computability
Lecture 20

Deduction theorem. For any σ-formulas φ, ψ and σ-theory T,

$$T, \varphi \vdash \psi \quad \text{iff} \quad T \vdash \varphi \to \psi.$$

Proof. $\Leftarrow$. Suppose $T \vdash \varphi \to \psi$. Then $T, \varphi \vdash \varphi$, hence by MP, $T, \varphi \vdash \psi$.

$\Rightarrow$. Suppose $T, \varphi \vdash \psi$ so there is a proof $(\psi_1, \psi_2, \dots, \psi_n)$ of ψ from T, φ. We show by induction on n that there is a proof of $\varphi \to \psi$ from T. Suppose for all $i < n$ there is a proof of $\varphi \to \psi_i$ from T and we show that $T \vdash \varphi \to \psi_n$.

Case 1: $\psi_n \in \text{Axiom}(\sigma) \cup T$. Then by (a) of previous proposition, $T \vdash \varphi \to \psi_n$.

Case 2: $\psi_n = \varphi$. Then by (b) of previous proposition, $\vdash \varphi \to \psi_n$, so $T \vdash \varphi \vdash \psi_n$.

Case 3: ψ_n is obtained from ψ_i, ψ_j by MP. Then $\psi_j = \psi_i \to \psi_n$ and by induction hypothesis, we have $T \vdash \varphi \to \psi_i$ and $T \vdash \varphi \to (\psi_i \to \psi_n)$. By Axiom (2),

$$\vdash (\varphi \to \psi_i) \to \big[(\varphi \to (\psi_i \to \psi_n)) \to (\varphi \to \psi_n)\big],$$

so by applying MP twice, we get $T \vdash \varphi \to \psi_n$. $\square$

Prop. Let φ, ψ be σ-formulas and v be a variable.

(a) $\vdash (\neg\neg\varphi) \to \varphi$

(b) $\vdash \varphi \to (\neg\neg\varphi)$

(c) $\vdash \varphi \to (\neg\varphi \to \psi)$ and $\vdash \neg\varphi \to (\varphi \to \psi)$

(We can abbreviate these as $(\varphi \wedge \neg\varphi) \to \psi$ and $(\neg\varphi \wedge \varphi) \to \psi$.)

(d) $\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$

(e) $\vdash \top$, where $\top := \forall v\,(v = v)$.

(f) $\vdash \bot \to \varphi$, where $\bot := \neg\top$

(g) $\vdash \varphi(t/v) \to \exists v\,\varphi$, where t is a σ-term that is OK to plug-in for v in φ.

Proof. (a) By Deduction, it's enough to prove $\neg\neg\varphi \vdash \varphi$.

(1) Axiom 3: $\vdash (\neg\varphi \to \neg\varphi) \to ((\neg\varphi \to \neg\neg\varphi) \to \varphi)$

(2) By (b) of prev. prop: $\vdash \neg\varphi \to \neg\varphi$.

(3) MP (2), (1): $\vdash (\neg\varphi \to \neg\neg\varphi) \to \varphi$.

(4) By (a) of prev. prop: $\neg\neg\varphi \vdash \neg\varphi \to \neg\neg\varphi$.

(5) MP (4), (3): $\neg\neg\varphi \vdash \varphi$.

(b) HW.

(c) By Deduction, it is enough to prove $\varphi, \neg\varphi \vdash \psi$. Axiom (3) gives $(\neg\psi \to \varphi) \to ((\neg\psi \to \neg\varphi) \to \psi)$, (a) of prev. prop gives $\varphi, \neg\varphi \vdash \neg\psi \to \varphi$ and $\varphi, \neg\varphi \vdash \neg\psi \to \neg\varphi$, and two applications of MP give $\varphi, \neg\varphi \vdash \psi$.

(d) By Deduction, it's enough to prove $\varphi \to \psi, \neg\psi \vdash \neg\varphi$. By Axiom (3), $(\neg\neg\varphi \to \psi) \to ((\neg\neg\varphi \to \neg\psi) \to \neg\varphi)$ and part (a) gives $\neg\neg\varphi \to \varphi$, then one can show using Axiom (2) that we get $\varphi \to \psi \vdash \neg\neg\varphi \to \psi$. Also, $\neg\psi \vdash \neg\neg\varphi \to \neg\psi$ by part (a) of prev. prop., so two MPs give $\varphi \to \psi, \neg\psi \vdash \neg\varphi$.

(e) By the equality axiom (6.a), $\vdash v=v$ so generalization axiom (5) gives $\vdash \forall v (v=v)$.

(f) We already have $\vdash \top$ and by (c), $\vdash \top \to (\bot \to \varphi)$, so MP gives $\vdash \bot \to \varphi$.

(g) $\exists v \varphi$ stands for $\neg \forall v \neg \varphi$, so by (d), we need to prove $\vdash \forall v \neg\varphi \to \neg\varphi(t/v)$. And this follows by instantiation axiom (4). $\square$

Constant Substitution Lemma. Let φ be a σ-formula in which v is a free variable, and let T be a σ-theory. Let c be a constant symbol that is not in σ. Then
$$T \vdash \varphi(c/v) \quad \text{iff} \quad T \vdash \varphi.$$

In other words, a new constant symbol has the same role as a free variable.

Proof. $\Leftarrow$. Suppose $T \vdash \varphi$. Then by generalization axiom (1'), $T \vdash \forall_v \varphi$, so instantiation gives $T \vdash \varphi(c/v)$.

$\Rightarrow$. Requires induction on the length of a proof of $\varphi(c/v)$ from T. This amounts to showing that if a formula $\varphi(c/v) \in \mathrm{Axiom}(\sigma \cup \{c\})$ then $\varphi \in \mathrm{Axiom}(\sigma)$, which one checks by hand, axiom-by-axiom. Left for HW. $\square$

Syntactic versions of consistency and completeness.

Def. Call a σ-theory T

- **consistent** if there is no σ-sentence φ such that $T \vdash \varphi$ and $T \vdash \neg\varphi$.
- **syntactically σ-complete** if for each σ-sentence φ, $T \vdash \varphi$ or $T \vdash \neg\varphi$.
- **σ-maximal consistent** if it is consistent and for each σ-sentence φ, $\varphi \in T$ or $\neg\varphi \in T$.

Prop. For a σ-theory T, the following are equivalent:

(1) T is consistent.

(2) $T \nvdash \bot$.

(3) $T \nvdash \varphi$ for some σ-sentence φ.

Proof. (2) $\Rightarrow$ (3). Trivial.

(1) $\Rightarrow$ (2). Because $T \vdash \top$, (1) says that T cannot prove $\neg\top =: \bot$.

(3) $\Rightarrow$ (1). We show $\neg(1) \Rightarrow \neg(3)$. Suppose $T \vdash \psi$ and $T \vdash \neg\psi$ for some σ-sentence ψ. Then for each σ-sentence φ, we have $T \vdash \psi \to (\neg\psi \to \varphi)$, so by two applications of MP, we get $T \vdash \varphi$. $\square$

Compactness for $\vdash$. For a τ-theory T and a τ-formula φ, if $T \vdash \varphi$ then $T_0 \vdash \varphi$ for some finite subtheory $T_0 \subseteq T$.

Proof. Proofs are finite! $\square$